Invariant measures for multidimensional diagonalizable group actions and arithmetic applications

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Introduction

Dynamical systems: Study of actions of (semi-)group H on a space X preserving some structure, e.g.

- measure / measure class / Borel structure,
- topology
- differential structure

Interesting case: actions that have complicated orbit structure (in paticular, no nice "X/H").

As in K. Schmidt's talk: We consider specific actions on concrete spaces preserving many types of structure, including a locally compact topology and a distinguished prob. measure. We will be especially interested in the case $H \cong \mathbb{R}^n$ or $H \cong \mathbb{Z}^n$ for $n \geq 2$ ("multiparameter ergodic theory").

Some general questions

1. How can orbits look like? in particular, what are the possibilities for $\overline{H.x}$ for $x \in X$?
2. What are the H-invariant probability measures on X?
3. How are the H-periodic orbits distributed in X? do they become equidistributed?

Notes:

- It is usually very easy to say how a.e. orbit behaves, we care about every orbit.
- If X not compact, we can also ask about locally finite H-invariant measures.
- For nonamenable H (e.g. free groups, groups with Kazhdan property T), classification of invariant measures seems less useful. Better question is classification of stationary measures[a].

[a] I.e. for a given probability measure ν on H whose support generates H, what are the probability measures μ on X satisfying $\nu * \mu = \mu$.

Flows on homogeneous spaces

Following is a general recipe for constructing interesting actions:

Ingredients:

- G is a nice (locally compact) group, e.g.
  - linear algebraic group — closed group of $n \times n$ matrices; in this talk: coefficients in $\mathbb{R}$ or $\mathbb{Q}_p$.
  - product of such.
  - also Lie groups, linear algebraic groups over $\mathbb{F}_p((t))$, etc.
- $\Gamma < G$ discrete subgroup
- $H < G$ unbounded ($\overline{H}$ not compact; usually H closed)

Recipe:

Take the action of $H < G$ on $\Gamma \backslash G$ by translations: $h.x = xh^{-1}$.

Variations: Instead of taking $H < G$ one can take H a subgroup of the affine transformations on G. This more general construction includes e.g. Furstenberg's action of $\times 2, \times 3$ on $\mathbb{R}/\mathbb{Z}$.

Remark: "Flow" typically refers to action of $\mathbb{R}$, but we will use it as a synonym to an action of arbitrary (nice) group.

Space of lattices

an important example of a $\Gamma\backslash G$

$G = \mathrm{SL}(n, \mathbb{R})$, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$
Then $X_n = \Gamma\backslash G \cong$ space of covolume one lattices in $\mathbb{R}^n$ by

$$\check{g} = \Gamma g \longleftrightarrow \text{lattice spanned by rows of } g$$

under this correspondence $g.\Lambda = \{vg^{-1} : v \in \Lambda\}$.
X_n has finite $\mathrm{SL}(n, \mathbb{R})$-invariant measure, but is not compact.

Sometimes better to think on X_n as space of lattices in $\mathbb{R}^n$ up to homothety (i.e. as $\mathrm{PGL}(n, \mathbb{Z})\backslash \mathrm{PGL}(n, \mathbb{R})$).

Mahler's criterion

$\{\Lambda_\alpha\}_\alpha$ is bounded iff $\exists \epsilon > 0$ so that for all α, every $v \in \Lambda_\alpha$ satisfies $\|v\| \geq \epsilon$.

Applications, particularly to number theory

Part of beauty of the subject: study of very concrete actions can have meaningful implications.

Example:

in late 1980's, Margulis proved long-standing Oppenheim conjecture by classifying bounded orbits of the group

$$H = \{h \in \mathrm{SL}(3, \mathbb{R}) \text{ preserving } Q(x, y, z) = xz + y^2\}$$

in space X_3 (space of lattices in $\mathbb{R}^3$ with $\mathrm{vol} = 1$) — a concrete action of a three-dimensional group on an eight-dimensional space.

Notation:

This group H is denoted by $\mathrm{SO}(2, 1)$.

Unipotent versus diagonalizable elements

G linear algebraic over k.

Definition

$g \in G$ is unipotent if 1 is the only eigenvalue of g over $\bar{k}$. $U < G$ is unipotent if every $u \in U$ is.

Examples:

$$u_1(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, u_2(t) = \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Note:

$U_2 = \{u_2(t)\}$ and its transpose generate $\mathrm{SO}(2,1)$.

Opposite extreme: diagonalizable groups over $\mathbb{R}$ (diagonalizable and not compact).

Dynamics of unipotent and diagonalizable flows

- Margulis: proved Oppenheim conjecture by studying dynamics of $H = \mathrm{SO}(2,1)$: a group generated by unipotents.
- Ratner: proved important general theorems about actions of such groups.
  - many number theoretic and other applications
- Diagonalizable flows: $\exists$ partial understanding (details later).
- ergodic theoretic (a.k.a. Kolmogorov-Sinai) entropy plays major role in the study of diagonalizable flows.

Preview of some applications of diagonal flows

We give a sample of what can be proved using current partial understanding of diagonal flows:

1. Littlewood conjectured (c. 1930) that for every $\alpha, \beta \in \mathbb{R}$,

(*) $$\inf_{n>1} n \|n\alpha\| \|n\beta\| = 0.$$

Einsiedler-Katok-L: the set of pairs (α, β) where (*) fails has Hausdorff dimension 0.

2. Conjecture (Rudnick and Sarnak, "Quantum Unique Ergodicity"): Let ϕ_i be a sequence of (normalized) Laplacian eigenfunctions on $\Gamma\backslash\mathbb{H}$. Then the probability measures $|\phi_i|^2 \, dm$ converge weakly to the uniform measure m.

L, Bourgain-L: True for Γ cocompact congruence-type lattices and the natural complete orthonormal sequence of eigenfuncs.

Remark: N. Ananthraman (Nonnenmacher) give (in particular) nontrivial bounds on dimension of limiting measure for general Γ by different methods.

Preview of some applications of diagonal flows
(continued)

3. Suppose K is a totally real number field (i.e. $[K:\mathbb{Q}] = n < \infty$ and all embeddings of K in $\mathbb{C}$ are real.)

Minkowski: Any ideal class $[I]$ of $\mathcal{O}_K$ has representative I with $N(I) = [\mathcal{O}_K : I] < O(\mathrm{disc}(K)^{1/2})$.
Likely sharp for $n = 2$ (sharp up to $\log \mathrm{disc}(K)^C$ for all n).

Einsiedler-L-Michel-Venkatesh: If $n > 2$, of the $\gg D^c$ fields K of $\mathrm{disc}(K) < D$ there are $\ll D^\epsilon$ which have an ideal class $[I]$ with $\min_{I \in [I]} N(I) \geq \epsilon \, \mathrm{disc}(K)^{1/2}$.

Unipotent flows

Ratner's theorems

Definition: H acts on X. $x \in X$ is H-periodic if the orbit $H.x$ supports an H-invariant probability measure.

Example: $H = \mathbb{Z}$: then x is periodic iff $n.x = x$ for some $n \in \mathbb{Z}$.

Definition

$X = \Gamma \backslash G$, μ a probability measure on X. μ is homogeneous (also: algebraic) if $\exists\, L < G$ so that μ is L-invariant probability measure on an L-periodic orbit.

Theorem (Ratner)

$X = \Gamma \backslash G$, $H < G$ generated by unipotents.

- *any H-invariant+ergodic[1] probability measure on X is homogeneous.*
- *for any $x \in X$, $\overline{H.x}$ is a periodic L-orbit for some $H \leq L \leq G$.*

[1] Invariant: $h.\mu = \mu$ for all $h \in H$; ergodic: $\nexists$ nonconstant H-inv. $f \in L^\infty(\mu)$.

Diagonalizable flows

Basic examples

Rank one case:

$G = \mathrm{SL}(2,\mathbb{R})$, $H = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$. Then H action on $\Gamma\backslash G$ "$\cong$" geodesic flow on $\Gamma\backslash\mathbb{H}$

- many possibilities for $\overline{H.x}$ and H-invariant measures (certainly need not be periodic/homogeneous respectively)

This is typical for action of 1-dimensional diagonalizable groups.

Higher rank case:

$X_n = \mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R}) \cong$ space of $\mathrm{vol} = 1$ lattices in $\mathbb{R}^n$.
$H =$ group of diagonal matrices with $\det = 1$.
For $n \geq 3$ ($\iff \dim H \geq 2$):

- Scarcity of orbit closures $\overline{H.x}$ and H-invariant measures conjectured by Furstenberg (unpublished), Katok-Spatzier, Margulis (scarcity of orbit closures implicitly conjectured by Cassels and Swinnerton-Dyer in 1955.)

Diagonalizable flows: invariant measures

Some conjectures (Furstenberg, Katok-Spatzier, Margulis)

Conjecture

$H =$ diagonal matrices, $X_n =$ space of $\mathrm{vol} = 1$ lattices in $\mathbb{R}^n$ for $n \geq 3$. Any H-invariant+ergodic probability measure on X_n is homogeneous.

Remark: H-invariant *homogeneous* probability measures on X_n easy to classify; in particular: either μ is H-periodic or $\operatorname{supp} \mu$ is unbounded.

General case: naive conjecture

For simplicity, take G linear algebraic $/\mathbb{R}$.

"Conjecture"

$H < G$ (closed) diagonalizable over $\mathbb{R}$, $\dim H \geq 2$. Let μ be H-invariant+ergodic probability measure on $\Gamma \backslash G$. Then

1. μ is homogeneous.

False! counter examples by M. Rees (already for $G = \mathrm{SL}(3, \mathbb{R})$, $H =$ diagonal matrices, $\Gamma < G$ cocompact).

Diagonalizable flows: invariant measures

Some conjectures (continued)

General case: corrected conjecture

G linear algebraic $/\mathbb{R}$. Γ discrete $+$ finite covolume (*lattice*).

Conjecture

$H < G$ (closed) diagonalizable over $\mathbb{R}$, $\dim H \geq 2$. Let μ be H-invariant$+$ergodic probability measure on $\Gamma\backslash G$. Then either

1. μ is homogeneous.
2. μ is supported on an L-periodic orbit ($H < L \leq G$) which has an algebraic rank one factor.

Algebraic rank one factor: suppose $\operatorname{supp}\mu \subset L.\check{g}$ ($\check{g} := \Gamma\backslash g$), and $\exists\phi : L \to F$ so that

- $\dim(\phi(H)) \leq 1$
- $\Gamma' = \phi(g^{-1}\Gamma g \cap L)$ discrete.

Then $\phi(H)$ acting on $\Gamma'\backslash F$ is a factor of action of H on $L.\check{g}$ with many invariant measures; each can be lifted to $\Gamma\backslash G$.

Ergodic theoretic entropy

brief introduction

All *theorems* about measures invariant under diagonalizable flows involve ergodic theoretic entropy, denoted by $h_\mu(g)$.

Proposition (entropy via dimension)

$H =$ diagonal matrices, $X_n =$ space of lattices,
μ an H-invariant$+$ergodic probability measure on X_n.
Then $h_\mu(g) > 0$ for some $g \in H$ iff

1. $\exists \delta > 0$ such that any measurable $S \subset X_n$ of Hausdorff dimension $< \dim(H) + \delta$ is μ-null.

This proposition is an example of more general relationship.
Alternative characterization via recurrence.

Ergodic theoretic entropy

brief introduction (continued)

Definition: Suppose U acts on X, μ probability measure (not necessarily U-invariant!). μ is U-recurrent (alternative terminology: conservative) if for every $B \subset X$ with $\mu(B) > 0$, for almost every $x \in B$, $\{u \in U : u.x \in B\}$ is unbounded.

Proposition (entropy via recurrence)

$H =$ diagonal matrices, $X_n =$ space of lattices, μ an H-invariant+ergodic probability measure on X_n. Then $h_\mu(g) > 0$ for some $g \in H$ iff any one of the following holds

1. μ is $U^+ = \begin{pmatrix} 1 & * & * \\ \vdots & \ddots & * \\ 0 & \dots & 1 \end{pmatrix}$-recurrent.
2. μ is recurrent under any Weyl-conjugate of U^+
3. μ is recurrent under some elementary 1-param unipotent subgroup U_{ij}.[2]

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[2] This uses Katok-Spatzier/Einsiedler-Katok/H. Hu

Invariant measures with positive entropy
(reminder)

Theorem (Einsiedler, Katok, L.)

$H =$ *diagonal matrices,* $X_n =$ *space of lattices for* $n \geq 3$*,* μ *an* H*-invariant+ergodic probability measure on* X_n*. Suppose* $h_\mu(g) > 0$ *for some* $g \in H$*. Then*

1. μ *is homogeneous, (not compactly supported).*

Return

Notes:

- in this case $\nexists$ periodic orbits with rank one factors because of global reasons (but this issue needs to be addressed in proof).
- Previous results by Katok-Spatzier, Einsiedler-Katok.
- Proof: combines two methods. "High entropy" [EK] when μ is recurrent for *many* elementary U_{ij}, and "low entropy" method [L, used to study QUE quetsion] using "dynamics" of μ along *one* U_{ij} (including techniques of Ratner).

Invariant measures with positive entropy
(continued)

Theorem (Einsiedler-L.)

G semisimple linear algebraic groups $/\mathbb{R}$ (not necessarily split), H maximal $\mathbb{R}$-diagonalizable subgroup, $\Gamma < G$ lattice. Suppose $h_\mu(g) > 0$ for some $g \in H$. Then at least one of the following two possibilities holds:

1. *There is some nontrivial semisimple ($\Leftarrow$ generated by unipotents) $L < G$ so that μ is L-invariant, and "L explains all the entropy of μ".*
2. *μ is supported on a periodic L-orbit with algebraic rank one factor (cf. ▸ Conjecture).*

Also: versions with G product of groups over $\mathbb{R}, \mathbb{Q}_p$; $\Gamma < G$ discrete not cocompact (but μ still a probability measure).
Proof follows general outline as in the previous Theorem, but there are substantial complications.

Values of products of linear forms

Notations, etc.: suppose $F(x_1, \ldots, x_n)$ is homogeneous polynomial of fixed degree. We will say that F is integral if it is *proportional* to some F with integer coefficients.

Conjecture (Cassels and Swinnerton-Dyer (1955), ?)

Let $F(x_1, x_2, \ldots, x_n)$ be a product of n linear forms in n variables[3] over $\mathbb{R}$ with $n \geq 3$. Assume F is not integral. Then

$$\inf_{0 \neq x \in \mathbb{Z}^d} |F(x)| = 0.$$

Cassels and Swinnerton-Dyer proved: Conjecture implies

Littlewood Conjecture (c. 1930)

For any $\alpha, \beta \in \mathbb{R}$, $\underline{\lim}_{n \to \infty} n \|n\alpha\| \|n\beta\| = 0$,
where $\|\cdot\| =$ distance to closest integer.

[3] We assume implicitly these linear forms are linearly independent.

Values of products of linear forms
(continued)

Definition: $F(x_1, \ldots, x_n)$ represents 0 nontrivially if $\exists$ nonzero $x \in \mathbb{Z}^n$ with $F(x) = 0$.

Remark: $n \|n\alpha\| \|n\beta\|$ is implicitly product of 3 linear forms in 3 variables evaluated at $\vec{n} \in \mathbb{Z}^3$; But: this form represents 0 nontrivially (hence C-SD Conjecture not directly applicable).

Let $\mathcal{F}_n =$ space of products of n linearly indep. forms in n vars. The map

$$g = (g_{ij}) \mapsto F_g = \prod_{k=1}^{n} (x_1 g_{1k} + \ldots x_n g_{nk})$$

is a surjective map $\mathrm{GL}(n, \mathbb{R}) \to \mathcal{F}_n$.
Let $\mathcal{F}_n^1$ be the image of $\mathrm{SL}(n, \mathbb{R})$ under this map; any $F \in \mathcal{F}_n =$ is proportional to a $F' \in \mathcal{F}_n^1$.
Note: For h diagonal with $\det h = 1$, F_g is the same as F_{gh}

Translation to diagonalizable flows

Consider property of $F \in \mathcal{F}_n^1$:

(*) $$\inf_{0 \neq x \in \mathbb{Z}^d} |F(x)| = 0$$

Note: $G = \mathrm{SL}(n, \mathbb{R})$ acts on $\mathcal{F}_n^1$ by linear change of variables. Property (*) is $\mathrm{SL}(n, \mathbb{Z})$-invariant under this action.

Conclusion: the statement "F_g has property (*)" depends only on $\Gamma g H$, i.e. on H orbit of $\check{g}$.

We need a dictionary

Γ-invariant properties of $F = F_g \in \mathcal{F}_n^1$	$\longleftrightarrow$	properties of H orbits $H.\check{g}$ in $\Gamma \backslash G$

An entry from dictionary forms↔flows

Proposition

F_g has property (*) iff $H.\check{g}$ is unbounded.

Proof: We show $H.\check{g}$ is unbounded ⇒ (*).

Skip proof

1. Mahler's criterion: since $H.\check{g}$ unbounded, for every $\delta > 0$ there is $h = \mathrm{diag}(h_1, \dots, h_n) \in H$ so that the lattice generated by rows of gh^{-1} contains nonzero vector v with $\|v\|_\infty \leq \delta$.
2. i^{th} coord. of v has form $(k_1 g_{1i} + \cdots + k_n g_{ni})h_i^{-1}$ for $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, and is $\underset{\text{in } |\cdot|}{\leqslant} \delta$.
3. since $\prod_i h_i = 1$, it follows that

$$F_g(k) = \prod_i (k_1 g_{1i} + \cdots + k_n g_{ni}) h_i^{-1} \underset{\text{in } |\cdot|}{\leqslant} \delta^n.$$

Since δ is arbitrary, F_g satisfies (*). □

More entries from dictionary

"F does not represent zero nontrivially" and "F integral" are also Γ-invariant properties, and we have

Proposition

F_g is both integral and does not represent zero nontrivially iff $H.\check{g}$ is periodic.

Hence C-SD Conjecture is equivalent to

Conjecture (C-SD;Margulis)

$n \geq 3$. For any $\check{g} \in X_n$, the orbit $H.\check{g}$ is either unbounded or periodic.

Note: false for $n = 2$.

Hausdorff dimension of exceptions to Littlewood and C-SD conjectures

Using "dictionary" and measure classification theorem for measures with positive entropy ▸ reminder:

Theorem (Einsiedler-Katok-L.)

$n \geq 3$. The set of $F \in \mathcal{F}_n^1$ for which $\inf_{0 \neq x \in \mathbb{Z}^d} |F(x)| > 0$ has zero Hausdorff dim.

Note: By C-SD conjecture, the set of "bad" F should be countable. Similarly,

Theorem (Einsiedler-Katok-L.)

The set of $(\alpha, \beta) \in \mathbb{R}^2$ for which $\underline{\lim}_{n \to \infty} n \|n\alpha\| \|n\beta\| > 0$ has zero Hausdorff dim.

Note:

- by Littlewood's conjecture, exceptional set should be $\emptyset$.
- it follows that for every α outside a set of Hausdorff $\dim = 0$, for every β, Littlewood holds (this strengthens result of Pollington-Velani).

Distribution of periodic H-orbits

Based on joint work with Einsiedler, Michel, Venkatesh

We want to study distribution of periodic H-orbits in X_n (more generally, periodic orbits of a maximal split Cartan on $\Gamma\backslash G$).

Basic Question: *to what extent do larger and larger periodic H-orbits fill out more and more of X_n?*

Some invariants of a periodic H-orbit $H.\check{g}$:

- shape of orbit, i.e. $\mathrm{stab}_H(\check{g}) = \{h \in H : h.\check{g} = \check{g}\}$.
- volume, i.e. $\mathrm{vol}(\mathrm{stab}_H(\check{g})\backslash H)$
- discriminant — an integer measuring "arithmetic complexity" of $H.\check{g}$ (not quite canonical)

Properties of discriminant and volume

1. $D^{c_1} \ll \#\{\text{periodic } H\text{-orbits with } \mathrm{disc} \leq D\} \ll D^{c_2}$ for some $c_1, c_2 > 0$.
2. $\log \mathrm{disc}(H.\check{g}) \ll \mathrm{vol}\, H.\check{g} \ll \mathrm{disc}(H.\check{g})^{c_3}$ ($\check{g}$ is H-periodic)

Distribution of individual orbits

C-SD/Margulis Conjecture implies the following about periodic orbits:

Conjecture

$n \geq 3$. Any compact $\Omega \subset X_n$ contains at most finitely many periodic H-orbits.

Using EKL classification theorem for measures with positive entropy, and separation properties of periodic H-orbits in terms of discriminants ("Linnik Principle") we prove:

Theorem (Einsiedler-Michel-Venkatesh-L.)

$n \geq 3$. *For any* $\epsilon > 0$ *and compact* $\Omega \subset X_n$*, the number of* H*-periodic orbits contained in* Ω *of discriminant* $\leq D$ *is* $\ll D^{\epsilon}$*.*

Notes

- Both Theorem and Conjecture fail for $n = 2$.
- Examples (suggested by Sarnak) show individual orbits do NOT need to equidistribute as the volume (equivalently disc.) $\to \infty$. This is in stark contrast to unipotent case.

Distribution of packets of periodic orbits

Even though individual periodic orbits may behave in strange ways, the collection of all periodic orbits of given *shape* and *discriminant* are much nicer (presumably, these collections often consist of a single orbit, but they can contain $\gg D^{1/2-\epsilon}$ orbits).

Theorem (Einsiedler-Michel-Venkatesh-L.)

Collections of all periodic orbits of given shape and discriminant in X_3 become equidistributed as the volume $\to \infty$.

Notes

- this is true even for $n = 2$, by theorem of Duke (special cases by Linnik/Skubenko).
- Proof combines EKL measure rigidity theorem with subconvex estimates on size of L-functions by Duke-Friedlander-Iwaniec used to establish positive entropy in the limit.
- Assuming GRH, this theorem holds for any prime n.

Open problems

$H =$ diagonal matrices, $X_n =$ space of lattices in $\mathbb{R}^n/$scalars for $n \geq 3$.

Conjecture 1

Any H-invariant$+$ergodic probability measure on X_n is homogeneous.

Conjecture 2

For any $\check{g} \in X_n$, the orbit $H.\check{g}$ is either unbounded or periodic.

Conjecture 3

Any compact $\Omega \subset X_n$ contains at most finitely many periodic H-orbits.

Open problems

(continued)

Conjecture 4

Fix $\rho > 0$. Let $H.\check{g}_i$ be a sequence of periodic H-orbits satisfying $\mathrm{vol}(H.\check{g}_i) \geq \mathrm{disc}(H.\check{g}_i)^\rho$. Then any weak limit of the corresponding probability measures is algebraic.

Note: in this case we allow also $n = 2$.

Also recall Furstenberg's conjecture:

Conjecture 5

The only nonatomic $\times 2$, $\times 3$-invariant measure on $\mathbb{R}/\mathbb{Z}$ is Lebesgue measure.

Problem 6

Do the preiodic orbits of the $\times 2$, $\times 3$-action on $\mathbb{R}/\mathbb{Z}$ become equidistributed?

Related question: how small can the group $\langle 2, 3 \rangle_{(Z/NZ)^\times}$ be for (6,N)=1? Can it be $O(\log N^2)$? Can it be $O(\log N^C)$?

Shana Tova
(Have a good year)