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Characterization of weak boundary values of L^p-functions by approximation

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Let $D \subset \mathbb{C}^n$ be bounded with C^1-boundary and $1 \leq p < \infty$. Then $f \in L^p(D)$ with $df \in L^p_1(D)$ has boundary values $f_b \in L^p(\partial D)$ such that Stokes Theorem is valid. If we just know $\overline{\partial} f \in L^p_{0,1}(D)$ we say that f has boundary values $f_b \in L^p(\partial D)$ if the Stokes Formula

$$\int_{\partial D} f_b \varphi|_{\partial D} = \int_D \overline{\partial} f \wedge \varphi + \int_D f \overline{\partial} \varphi \tag{1}$$

holds for all $\varphi \in C^\infty(\overline{D})$. Such boundary values play a decisive role in the study of the boundary regularity of the $\overline{\partial}$-equation or the complex Green operator. In this talk we show that the space of functions with such L^p-boundary values is exactly the completion of $C^\infty(\overline{D})$ under the norm

$$\|f\|_* = \|f\|_{L^p(D)} + \|\overline{\partial} f\|_{L^p_{0,1}(D)} + \|f|_{\partial D}\|_{L^p(\partial D)}.$$

Similar results are true for forms of higher degree. As applications, we show that $f \in L^1_{loc}(D)$ with $\overline{\partial} f = 0$ in the sense of distributions is C^∞-smooth and that Stokes Formula (1) holds for $f \in C^0(\overline{D})$ with $\overline{\partial} f \in L^1_{0,1}(D)$ (in that case $f_b = f|_{\partial D}$).

Setting

$D \subset\subset \mathbb{C}^n$, ∂D C^1**-smooth,** $\iota : \partial D \hookrightarrow \mathbb{C}^n$, $1 \leq p < \infty$**.**

Definition

Let $f \in L^p(D)$ **with** $\overline{\partial} f \in L^p_{0,1}(D)$**.**
Then f **has got boundary values** $f_b \in L^p(\partial D)$ **if**

$$\int_{bD} f_b \cdot \iota^*(\varphi) = \int_D \overline{\partial} f \wedge \varphi + \int_D f \cdot \overline{\partial} \varphi \tag{2}$$

for all $\varphi \in C^\infty_{n,n-1}(\overline{D})$**. Let** $B^p(D)$ **be the space of all such functions.**

Where do such boundary values appear?

Example:

Theorem (Harvey/Polking 1984)

Let r **be the strictly plurisubharmonic defining boundary function of a strictly pseudoconvex domain** $D \subset\subset \mathbb{C}^n$ **and** $f \in L^1_{0,1}(D)$ **with** $\overline{\partial} f = 0$ **and**

$$|r|^{-1/2} \overline{\partial} r \wedge f \in L^1_{0,2}(D).$$

Then there exists $u \in B^1(D)$ **such that**

$$\overline{\partial} u = f.$$

Denote the completion of

$$C^\infty(\overline{D})$$

under the norm

$$\|f\|_{*,p} = \|f\|_{L^p(D)} + \|\overline{\partial} f\|_{L^p_{0,1}(D)} + \|f|_{\partial D}\|_{L^p(\partial D)}$$

by

$$\widehat{C^\infty_p}(D) = \big(\widehat{C^\infty(\overline{D}), \| \cdot} \|_{*,p}\big).$$

Then our result (in the case of functions) is:

<u>Theorem (R., 2006)</u>
Let $1 \leq p < \infty$**. Then:**

$$\widehat{C^\infty_p}(D) = B^p(D).$$

<u>Proof:</u> The main idea is: convolution with a carefully choosen Dirac-sequence. □

Similar results are true for forms of higher degree.

Remark on continous boundary values

Let $f \in C^0(\overline{D})$**,** $\overline{\partial} f \in L^1_{0,1}(D)$**.**
Then:

$$f^j|_{\partial D} \to f|_{\partial D} \textbf{ in } L^\infty(\partial D)$$

$\Rightarrow f \in B^1(D)$ **with** $f_b = f|_{\partial D}$**.**

Applications:

I. Bochner-Martinelli-Koppelman-Formula (for L^p-functions)

Let $f \in C^1(\overline{D})$**. Then**

$$f(z) = \int_{\partial D} f|_{\partial D}(\zeta) K_0(\zeta, z) - \int_D \overline{\partial}_\zeta f(\zeta) \wedge K_0(\zeta, z) \quad (3)$$

with

$$K_0(\zeta, z) = \frac{1}{(2\pi i)^n} \beta^{-n} \partial\beta \wedge \left(\overline{\partial}\partial\beta\right)^{n-1}, \; \beta = \|\zeta - z\|^2.$$

Remarks:

1. $K_0 \in C^\infty(\mathbb{C}^n \times \mathbb{C}^n \setminus \{\zeta = z\})$

2. $g \mapsto \int_{\partial D} g K_0$ **is continous** $L^1(\partial D) \to L^1(D)$

3. $g \mapsto \int_D g \wedge K_0$ **is cont.** $L^1_{0,1}(D) \to L^1(D)$

$\Rightarrow$ (3) **is valid in** $L^1(D)$ **for** $f \in B^1(D)$**.**

II. Regularity of holomorphic functions (a)

Let $f \in B^1(D)$**,** $\overline{\partial} f = 0$**.**
BMK-Formula:

$$f(z) = \int_{\partial D} f_b(\zeta) K_0(\zeta, z) \quad \textbf{f. a. a. } z \in D.$$

$\Rightarrow f \in C^\infty(D)$**.**

III. Regularity of holomorphic functions (b)

Let $G \subset \mathbb{C}^n$ **open,** $f \in L^1_{loc}(G)$**,** $\overline{\partial} f = 0$**.**
Then: $f \in B^1(B_\epsilon(p))$ **f. a. a.** $B_\epsilon(p) \subset\subset G$
$\Rightarrow f \in C^\infty(G)$**.**

Proof: **Let** $p \in G$ **and** $\delta > 0$ **such that**

$$B_\delta(p) \subset\subset G.$$

Then there exists $\{f^j\}_{j=1}^\infty \subset C^\infty(B_\delta(p))$ **such that**

$$\begin{aligned} f^j &\rightarrow f \quad \textbf{in } L^1(B_\delta(p)), \\ \overline{\partial} f^j &\rightarrow 0 = \overline{\partial} f \quad \textbf{in } L^1_{0,1}(B_\delta(p)). \end{aligned}$$

Fubini $\Rightarrow$ $f^j|_{\partial B_\epsilon(p)}$ **converges in** $L^1(\partial B_\epsilon(p))$ **for almost all** $0 < \epsilon < \delta$**.**

$\Rightarrow f \in B^1(B_\epsilon(p))$ **f. a. a.** $0 < \epsilon < \delta$**.**

The last step follows with II. $\square$