On the domain of hypoelliptic Ornstein-Uhlenbeck operators

Bálint Farkas (Darmstadt)

joint with

Alessandra Lunardi (Parma)

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Hypoelliptic Kolmogorov (Ornstein-Uhlenbeck) operators

- $u : \mathbb{R}^d \to \mathbb{R}$, $x \in \mathbb{R}^d$

$$\mathcal{L}u(x) = \frac{1}{2}\sum_{i,j=1}^{d} q_{ij} D_{ij} u(x) + \sum_{i,j=1}^{d} b_{ij} x_j D_i u = \frac{1}{2}\operatorname{Tr}(Q D^2 u(x)) + \langle Bx, Du(x)\rangle$$

- Q, B $n \times n$ matrices, $Q \geq 0$, Q possibly degenerate
- Define $Q_t := \int_0^t e^{sB} Q e^{sB^*}\, \mathrm{d}s$
- $\mathcal{L}$ is hypoelliptic if $\det Q_t \neq 0$ for all/some $t > 0$
- $\mathcal{N}_{e^{tB}x, Q_t}$ Gaussian measures with
  - covariance operator Q_t
  - mean $e^{tB}x$
  - all are absolutely continuous with respect to the d-dimensional Lebesgue measure

density: $$\frac{1}{(2\pi)^{d/2}(\det Q_t)^{1/2}} e^{-\frac{1}{2}\langle Q_t^{-1}(e^{tB}x - \cdot), e^{tB}x - \cdot\rangle}$$

The Kalman rank condition

Hypoellipticity

$\Updownarrow$

Kalman rank condition

$$[Q^{1/2}, BQ^{1/2}, B^2Q^{1/2}, \ldots, B^{n-1}Q^{1/2}] \quad \text{has rank } d\text{, for some } n \le d$$

Take n the smallest possible

Define

- $V_h := \operatorname{Range} Q^{1/2} + \operatorname{Range} BQ^{1/2} + \cdots + \operatorname{Range} B^hQ^{1/2}$ for $h = 0, \ldots, n-1$
- $W_0 := V_0$, $W_h := V_h \ominus V_{h-1}$ if $h = 1, \ldots, n-1$

then $\mathbb{R}^d = W_0 \oplus W_1 \oplus \cdots \oplus W_{n-1}$

take ONB in $W_h \longrightarrow$ ONB $\{e_1, \ldots, e_d\}$ in $\mathbb{R}^d$

for $h = 0, \ldots, n-1$ let I_h the set of indices i with $e_i \in W_h$

Infinitesimally invariant measures

A μ Borel probability measure on $\mathbb{R}^d$ is infinitesimally invariant if

$$\int_{\mathbb{R}^d} \mathcal{L}u \,\mathsf{d}\mu = 0$$

for all $u \in C_b^2(\mathbb{R}^d)$

- For $u \in C_b^2(\mathbb{R}^d)$: $\qquad \mathcal{L}(u^2) = 2u\mathcal{L}u + \langle QDu, Du \rangle$
- Integrating with respect to μ

$$0 = \int_{\mathbb{R}^d} \mathcal{L}(u^2)\,\mathsf{d}\mu = \int_{\mathbb{R}^d} 2u\mathcal{L}u\,\mathsf{d}\mu + \int_{\mathbb{R}^d} \mathcal{L}(u^2)\langle QDu, Du\rangle\,\mathsf{d}\mu$$

$$\Downarrow$$

$$\langle \mathcal{L}u, u \rangle_{L^2(\mathbb{R}^d,\mu)} = \int_{\mathbb{R}^d} u\,\mathcal{L}u\,\mathsf{d}\mu = -\frac{1}{2}\int_{\mathbb{R}^d} \langle QDu, Du\rangle\,\mathsf{d}\mu \leq 0$$

- $\implies \mathcal{L} : D(\mathcal{L}) := C_b^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d, \mu)$ is dissipative
- $(L, D(L))$ is its closure in $L^2(\mathbb{R}^d, \mu)$

An example

A special case is:

$$\begin{aligned}\mathcal{L}u(x,y) &= \frac{1}{2}u_{xx}(x,y) - (x+y)u_x(x,y) + xu_y(x,y)\\ &= \frac{1}{2}\operatorname{Tr}(QD^2u(x,y)) + \langle B(x,y)^\top, Du(x,y))\rangle\end{aligned}$$

- $Q = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, $B = \begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}$, $[Q^{1/2}, BQ^{1/2}] = \begin{pmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}$
- $n = 2$, $W_0 = \mathbb{R} \times \{0\}$, $W_1 = \{0\} \times \mathbb{R}$
- $e_1 = (1,0)^\top$, $e_2 = (0,1)^\top$, $I_0 = \{1\}$, $I_1 = \{2\}$

The invariant measure has density with respect to the Lebesgue measure λ:

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x,y) = e^{-(x^2+y^2)}$$

Maximal regularity

Question.

- If $g : \mathbb{R}^d \to \mathbb{R}$ and $\lambda f - \mathcal{L}f = g$, $\lambda > 0$ which regularity properties does f have?

Answer 1 Q *invertible:*

- $L^2(\mathbb{R}^d, \mu)$ *setting:* $D(L) = H^2(\mathbb{R}^d, \mu)$

  A. Lunardi, G. Da Prato

- $L^p(\mathbb{R}^d, \mu)$ *setting:* $D(L) = W^{2,p}(\mathbb{R}^d, \mu)$

  G. Metafune, J. Prüß, A. Rhandi, R. Schnaubelt

Answer 2 *Special case*

$$\mathcal{L}u(x, y) = \frac{1}{2}u_{xx}(x, y) - (x + y)u_x(x, y) + xu_y(x, y)$$

- $g \in L^2_{loc}(\mathbb{R}^2)$, $\lambda f - \mathcal{L}f = g \Longrightarrow f \in H^{2,2/3}_{loc}(\mathbb{R}^d)$

  E. M. Stein, L. Rothschild, G. B. Folland

- $C_b(\mathbb{R}^d)$ *setting:* $D(L) \subseteq C_b^{2,2/3}(\mathbb{R}^d)$

  A. Lunardi

The Ornstein–Uhlenbeck semigroup

For $f \in L^2(\mathbb{R}^d, \mu)$, $x \in \mathbb{R}^d$ define

$$(T(t)f)(x) := \int_{\mathbb{R}^d} f \,\mathsf{d}\mathcal{N}_{e^{tB}x,Q_t} = \frac{1}{(2\pi)^{d/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^d} e^{-\frac{1}{2}\langle Q_t^{-1}y,y\rangle} f(e^{tB}x - y)\,\mathsf{d}y$$

Theorem. The operators $T(t)$ are

- contractive,
- positivity preserving,
- and form a strongly continuous semigroup on $L^2(\mathbb{R}^d, \mu)$
  - $T(0) = \mathrm{Id}$, $T(t+s) = T(t)T(s)$ for $t \geq 0$
  - $t \mapsto T(t)f$ is continuous for all $f \in L^2(H, \mu)$
- whose infinitesimal generator is $(L, D(L))$

The main result I.

Theorem. For

$$\mathcal{L}u(x,y) = \frac{1}{2}u_{xx}(x,y) - (x+y)u_x(x,y) + xu_y(x,y)$$

we have $D(L) \subseteq H^{2,2/3}(\mathbb{R}^d, \mu)$

For the proof:

- An abstract interpolation result
- Gradient estimates
- Weighted, anisotropic, fractional Sobolev spaces

Theorem. [A. Lunardi]

- X, E Banach spaces, $E \subseteq X$
- $(T(t))_{t\geq 0}$ a C_0-sgrp. in X, generator $(A, D(A))$
- $m \in \mathbb{N}$ and $\exists\ 0 < \beta < 1$, $\omega \in \mathbb{R}$, $c > 0$ such that

$$\|T(t)\|_{\mathcal{L}(X,E)} \leq \frac{ce^{\omega t}}{t^{m\beta}} \qquad \text{for } t > 0$$

Then

$$(X, D(A^m))_{\theta,p} \subset (X, E)_{\theta/\beta,p} \qquad \text{for all } \theta \in (0, \beta) \text{ and } 1 \leq p \leq \infty$$

Gradient estimates

Lemma. For $\omega > 0$ there exist a constant c such that

$$\|D_x^6 T(t) f\|_2 \leq \frac{ce^{\omega t}}{t^3}\|f\|_2, \qquad \|D_y^2 T(t) f\|_2 \leq \frac{ce^{\omega t}}{t^3}\|f\|_2$$

for all $t \in (0, +\infty)$, $f \in L^2(\mathbb{R}^2, \mu)$.

This means for $f \in L^2(\mathbb{R}^2, \mu)$, $t > 0$

$$T(t)f \in H^{6,2}(\mathbb{R}^2, \mu) \quad \text{and} \quad \|T(t)\|_{\mathcal{L}(L^2, H^{6,2})} \leq \frac{ce^{\omega t}}{t^3}$$

$\Longrightarrow$ the Lemma is applicable

- $X = L^2(\mathbb{R}^2, \mu)$, $E = H^{6,2}(\mathbb{R}^2, \mu)$
- take $m = 4$, $\theta = 1/4$, $\beta = 3/4$, $p = 2$

$$(L^2, D(L^4))_{1/4,2} \subseteq (L^2, H^{6,2})_{1/3,2}$$

- $(L^2, D(L^4))_{1/4,2} = D(L)$, and $(L^2, H^{6,2})_{1/3,2} = H^{2,2/3}(\mathbb{R}^2, \mu)$

  so $D(L) \subseteq H^{2,2/3}(\mathbb{R}^2, \mu)$

The main result II.

Theorem. [B. Farkas, A. Lunardi]

Let $k \in \mathbb{N}$. For the domain of the Ornstein–Uhlenbeck operator L we have

$$D(L^k) \subseteq H^{2k,\, 2k/3,\, 2k/5, \dots,\, 2k/(2n-1)}(\mathbb{R}^d, \mu)$$

$$D(L) \subseteq H^{2,\, 2/3,\, 2/5, \dots,\, 2/(2n-1)}(\mathbb{R}^d, \mu)$$

- $g \in L^2_{loc}(\mathbb{R}^2)$, $\lambda f - \mathcal{L} f = g \Longrightarrow f \in H^2_{loc}(\mathbb{R}^d)$ w.r.t. I_0
  G. B. Folland
- $g \in L^2_{loc}(\mathbb{R}^2)$, $\lambda f - \mathcal{L} f = g \Longrightarrow f \in H^{2/(2n-1)}_{loc}(\mathbb{R}^d)$ w.r.t. I_{n-1}
  E. M. Stein, L. Rothschild
- $C_b(\mathbb{R}^d)$ setting: $D(L) \subseteq C_b^{2,2/3,2/5,\dots,2/(2n-1)}(\mathbb{R}^d)$
  A. Lunardi